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THE INFLUENCE OF A LIMITING DISLOCATION FLUX ON THE MECHANICAL RESPONSE OF POLYCRYSTALLINE METALS

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Abstract—Certain physical results from the study of dislocation motions lead to the concept of ideal viscoplasticity. In particular the existence of a limiting dislocation velocity, coupled with an upper limit on dislocation density, provides an upper bound on the dislocation flux. Thus, the plastic strain rate in polycrystalline metals must also be bounded in many cases of interest. This physical situation can be idealized by postulating transition functions from zero to maximum flux in as simple as possible a manner consistent with the problem under investigation and the solution features to be examined. A drastic transition function is given here which leads to multiaxial stress, strain, strain-rate relations of reasonable simplicity and these are illustrated by application to several example problems. A common feature of the solutions of the examples treated is that the material response is partly rate dependent and partly rate independent. This indicates that the corresponding physical situations are characterized by large dislocation fluxes during part of the time and very small fluxes at other times.

1. INTRODUCTION

The concept of dislocations as the most important mechanism for plastic deformation of crystalline solids arose as attempts were made to reconcile the low yield strengths observed in crystalline materials with the high theoretical strengths suggested by Frenkel[1]. The theory of crystal dislocations at present is still concerned with many pseudo-static problems such as dislocation models of microcracks and grain boundaries, and theories of work hardening. However, an area in which dislocation theory has been particularly fruitful is the description of rate sensitivity or, more generally, time dependent mechanical behavior of solids. It is this area that is examined in the present paper. Historically, the major insights to understanding pseudo-static plastic deformation behavior were provided by the concept of ideal plasticity. This gross analytical simplification of the complex response of a ductile, polycrystalline metal, coupled with astute experimentation, has been the most important factor in the development of our understanding of macroscopic plastic behavior. To provide a comparable simplification of the complex response of dislocations suggested by various

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theories and many illuminating experiments the concept of ideal viscoplasticity was proposed[2].

This concept can best be described by analogy with ideal plasticity. The essential physical facts incorporated into an ideally plastic formulation are that the material response is partly elastic and partly plastic. The description of the elastic portion is always quite simple, either Hookean or rigid, while the plastic description is taken to be as simple as is consistent with the features to be examined in the solution. The essential physical facts incorporated into an ideally viscoplastic formulation are that the dislocation flux, which determines the plastic deformation rate, is (approximately) zero at sufficiently low loads and is bounded above even for very high loads. The upper bound on dislocation flux derives from the two conditions that dislocation density ultimately reaches a saturation value and that dislocation velocity can not exceed the shear wave velocity in the material. Under most test conditions velocity tends to change much more rapidly than density and so it is usually the velocity limitation that is most important physically. An ideally viscoplastic description of these facts postulates a dislocation flux, and, hence, a plastic strain rate, having a transition from zero to a limiting value. As in ideal plasticity, the transition relation is taken to be as simple as is consistent with the features to be examined in the solution.

In the original paper[2] the examples used to illustrate the concept of ideal viscoplasticity were extremely simple and were motivated by arguments based mainly on dislocation theory. In the present paper a three-dimensional formulation is presented based mainly on analogy with ideal plasticity and examples are given in which the theory is applied to multiaxial stress problems and to one problem of wave propagation.

The procedure here is open to the immediate criticism that strain rates corresponding to the maximum possible dislocation flux seldom occur in real materials. This is again analogous to the criticism of a perfectly plastic idealization of material behavior. In the latter case the selection of a value for the yield stress must be governed by the strain rates expected in a particular problem, for the results of calculations to have quantitative significance. This point is discussed further in section 6.

2. IDEAL PLASTICITY

The constitutive equations of the classical theory of plasticity can be motivated in their development in a number of alternative ways. It is usual to lay emphasis on the existence of a yield function, of a flow potential and on the postulate that the model be rate independent. It is also possible to develop these equations by considering them to be the limit of a set of viscoplastic material models. This is done in, for example, the basic text by Prager[3] where the stress–strain relations of the von Mises material are derived from those of the Bingham material. A further approach is to obtain a plasticity relationship from a family of creep laws in which the inelastic shear strain rate $\dot{\gamma}^p$ is related to the applied shear stress τ through the equation

$$\dot{\gamma}^p = (1/\beta)(\tau/\tau_0)^n \tag{2.1}$$

where β, a characteristic time, τ_0, a characteristic stress and n, are material constants. For various values of n this equation gives a family of creep laws which tend with increasing n to the rate independent, perfectly plastic material.

Multiaxial versions of this model are possible and an example defining the response of an isotropic material for which the creep is isochoric and independent of pressure relates the deviatoric strain rate $\dot{e}_{ij}$ to the deviatoric stress s_{ij} by the equation

$$\dot{e}_{ij} = \dot{s}_{ij}/2G + (1/2\beta)[(\tfrac{1}{2}s_{kl}s_{kl})^{1/2}/\tau_0]^n[s_{ij}/(\tfrac{1}{2}s_{pq}s_{pq})^{1/2}] \tag{2.2}$$

where β, τ_0 and n are as before and G is the shear modulus of the material.

In the limit as $n \to \infty$ this model predicts

$$\dot{e}_{ij} = \dot{s}_{ij}/2G + \begin{cases} 0; & \tfrac{1}{2}s_{ij}s_{ij} < {\tau_0}^2 \\ \lambda s_{ij}; & \tfrac{1}{2}s_{ij}s_{ij} = {\tau_0}^2 \end{cases} \tag{2.3}$$

where λ is an indeterminate constant which serves as a factor of proportionality. When G is finite this is the Prandtl–Reuss model and when $G \to \infty$ the Huber–Mises model.

3. DISLOCATION THEORY

A material model of the foregoing kind has a counterpart in the physical theory of materials. A basic equation relating the macroscopic plastic flow to the internal structure of a material[4] is

$$\dot{\gamma}^p = bNv \tag{3.1}$$

where $\dot{\gamma}^p$ is as before, the Burgers vector b is a characteristic length, N is the dislocation density in length of line per unit volume and v is the velocity of dislocation motion. In a seminal paper Johnston and Gilman[5] reported experimental measurements of dislocation velocities which demonstrated that the velocities were mainly dependent on stress level. They found that the phenomenon could be accurately represented by the formula

$$v = v_0(\tau/\tau_0)^n \tag{3.2}$$

where v_0 is the velocity corresponding to a stress τ_0 in the material. For the material on which their experiments were carried out, namely LiF, the value of n was 16. Many other materials have been examined in the same way and the values of the exponents for these materials are given in Table 1 taken from Hall[6]. It is clear that for a large number of

Table 1. Stress exponent for dislocation velocity (Values at room temperature)

Crystal class	Material	Stress exponent
b.c.	Fe (high purity)	5–6
	Fe (0·05% C)	15
	Fe–Si	20
	Fe–Si	35–40
	Fe–Si	44
	Fe–Si	~60
	Cr	5–9
	Nb	7–18
	Mo	6·4
	W	4·8
	NiAl	~40
f.c.c.	Cu	0·7
	Pb	1·0
	Al	1·0
	Ni	16
Cubic	LiF	16–25
	NaCl	0·8
h.c.p.	Zn	1·0

Adapted from Hall[5].

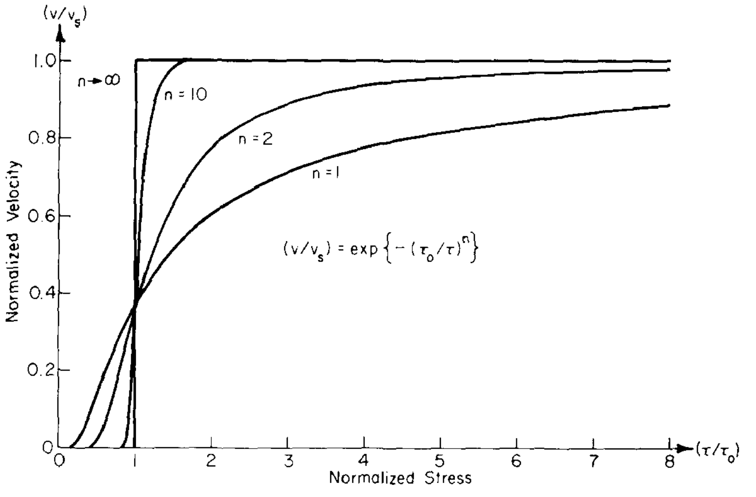

Fig. 1. An abrupt transition of dislocation velocity with stress derived from the relation $(v/v_s) = \exp\{-(\tau_0/\tau)^n\}$ in the limit as n grows increasingly large.

metals the exponent n is large which explains the success of the classical theory of plasticity in predicting rate independent behavior in metals.

The above formula predicts that as the stress increases the velocity increases without limit. However there are theoretical reasons,[4], to believe that the dislocation velocity is bounded above by the speed of shear waves in the material. Reflecting this idea a velocity function was proposed by Gilman[7] of the form

$$v = v_s \exp\{-\tau_0/\tau\} \tag{3.3}$$

where v_s is the limiting dislocation velocity. A modification of this equation which corresponds with the family of models of equation (3.2) is

$$v = v_s \exp\{-(\tau_0/\tau)^n\}. \tag{3.4}$$

As shown in Fig. 1 the limit of (3.4) as $n \to \infty$ is a velocity, stress relation of the form

$$\begin{aligned} v &= 0; & \tau &< \tau_0 \\ 0 < v &< v_s; & \tau &= \tau_0 \\ v &= v_s; & \tau &> \tau_0 . \end{aligned} \tag{3.5}$$

Equation (3.5) emphasize the fact that bNv_s is an upper bound to the plastic shear strain rate.

4. IDEAL VISCOPLASTICITY

A simple multiaxial constitutive equation relating deviatoric stress and strain and corresponding to the foregoing uniaxial model is

$$\dot{e}_{ij} = \dot{s}_{ij}/2G + (1/2\beta)\exp\{-[\tau_0/(\tfrac{1}{2}s_{kl}s_{kl})^{1/2}]^n\}s_{ij}/(\tfrac{1}{2}s_{pq}s_{pq})^{1/2}. \tag{4.1}$$

Where $(1/\beta)$ is now identified with bNv_s. In the limit as $n \to \infty$ this predicts

$$\dot{e}_{ij} = \dot{s}_{ij}/2G + \begin{cases} 0; & \tfrac{1}{2}s_{ij}s_{ij} < \tau_0^2 \\ \lambda s_{ij}; & \tfrac{1}{2}s_{ij}s_{ij} = \tau_0^2 \\ s_{ij}/[2\beta(\tfrac{1}{2}s_{kl}s_{kl})^{1/2}]; & \tfrac{1}{2}s_{ij}s_{ij} > \tau_0^2 \end{cases} \tag{4.2}$$

where, as before, λ is a factor of proportionality. The effect of the limiting dislocation velocity coupled with the notion of an ideal material is to produce a material which is rate independent for certain rates of loading and rate dependent when the rate is high enough to produce stresses above a particular level. If we define an equivalent plastic shear strain rate through

$$\dot{\gamma}_{eq}{}^{p} = 2\sqrt{(\tfrac{1}{2}\dot{e}_{ij}{}^{p}\dot{e}_{ij}{}^{p})}; \quad \dot{e}_{ij}{}^{p} = \dot{e}_{ij} - \dot{s}_{ij}/2G \tag{4.3}$$

then $1/\beta$ is the upper bound to this strain rate.

5. EXAMPLES

The concept of ideally viscoplastic material response is illustrated here by application of (4.2) to three examples which correspond to not infrequently encountered test situations. Bear in mind that (4.2) represents a particular postulate and that other idealizations of greater or less complexity could be taken according to the requirements of a given analysis. An important feature of the results obtained here is that the material response is rate sensitive during only a part of the time it is under load, namely those times for which the stress state exceeds the critical value prescribed by (4.2).

(i) *Constant strain*

Consider that a constant deviatoric strain, $e_{ij} = e_{ij}{}^{*}$, is imposed for $t \geq 0$, for a material unstressed and unstrained for $t < 0$, such that

$$2G(\tfrac{1}{2}e_{ij}{}^{*}e_{ij}{}^{*})^{1/2} > \tau_0. \tag{5.1.1}$$

The appropriate equation for t in the neighborhood of $t = 0$ is

$$\dot{s}_{ij}/2G + s_{ij}/[2\beta(\tfrac{1}{2}s_{kl}s_{kl})^{1/2}] = 0 \tag{5.1.2}$$

with the initial condition $s_{ij}(0) = s_{ij}{}^{*} = 2Ge_{ij}{}^{*}$. The solution of this equation is

$$s_{ij}(t)/2G = [(\tfrac{1}{2}s_{pq}{}^{*}s_{pq}{}^{*})^{1/2}/2G - t/2\beta)]s_{ij}{}^{*}/(\tfrac{1}{2}s_{kl}{}^{*}s_{kl}{}^{*})^{1/2} \tag{5.1.3}$$

valid for

$$t/2\beta \leq [(\tfrac{1}{2}s_{ij}{}^{*}s_{ij}{}^{*})^{1/2} - \tau_0]/2G \tag{5.1.4}$$

and for $t/2\beta$ greater than this, no further relaxation of stress takes place. This response is indicated in Fig. 2 where the linear decay with time of the stress invariant $(\tfrac{1}{2}s_{k}\cdot s_{kl})^{1/2}$ is shown for three different initial stress levels. As shown above, each individual stress component decreases similarly but from the individual initial levels to individual steady-state values proportional to their initial levels.

(ii) *Tension–Torsion*

To further illustrate the dual nature of this material model let us consider the case of a thin walled tube loaded in simultaneous tension and torsion. The tension stress and strain we will denote by σ and ε and the torsional stress and strain by τ and γ. For an incompressible material, assumed here for simplicity, the appropriate equations are

$$\dot{\varepsilon} = \dot{\sigma}/3G + \begin{cases} 0; & (\sigma^2/3 + \tau^2) < \tau_0{}^2 \\ 2\lambda\sigma/3; & (\sigma^2/3 + \tau^2) = \tau_0{}^2 \\ \sigma/[3\beta(\sigma^2/3 + \tau^2)^{1/2}]; & (\sigma^2/3 + \tau^2) > \tau_0{}^2 \end{cases} \tag{5.2.1}$$

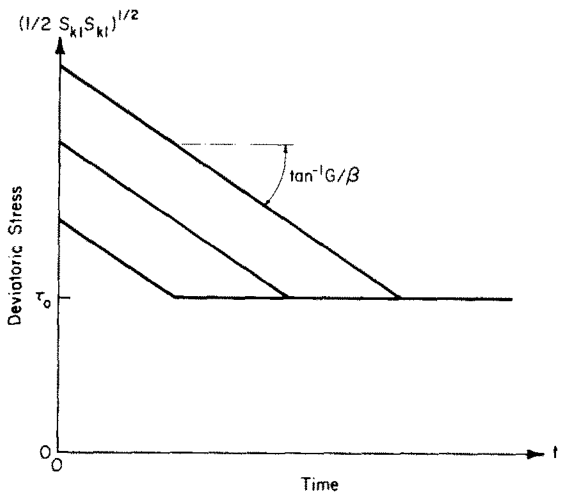

Fig. 2. Solution to the problem of instantaneous strains imposed at time zero and held fixed, based on the ideal viscoplastic material specified in text.

$$\dot{\gamma} = \dot{\tau}/G + \begin{cases} 0; & (\sigma^2/3 + \tau^2) < {\tau_0}^2 \\ 2\lambda\tau; & (\sigma^2/3 + \tau^2) = {\tau_0}^2 \\ \tau/[\beta(\sigma^2/3 + \tau^2)^{1/2}]; & (\sigma^2/3 + \tau^2) > {\tau_0}^2 \end{cases} \tag{5.2.2}$$

and the proportionality factor λ where applicable is given by

$$\lambda = (\sigma\dot{\varepsilon} + \tau\dot{\gamma})/2{\tau_0}^2. \tag{5.2.3}$$

For the purpose of this illustrative example we take the tube to be in tension at the yield point at $t = 0$ and for $t > 0$ to be twisted at a specified rate with the extension held fixed. Thus

$$\begin{aligned} &\sigma(0) = \sqrt{3}\,\tau_0, \qquad \varepsilon(0) = \sqrt{(3)}\,\tau_0/3G \\ &\tau(0) = 0, \qquad \gamma(0) = 0 \end{aligned} \tag{5.2.4}$$

and

$$\dot{\varepsilon} = 0, \qquad \dot{\gamma} = \alpha/\beta;\ \alpha > 1 \quad \text{for} \quad t > 0. \tag{5.2.5}$$

The corresponding problem for the incompressible Prandtl–Reuss material has been given in considerable detail by Prager and Hodge[8] and it will be our purpose here to give a comparison result which occurs when a limiting velocity is imposed to dislocation motion and consequently to the plastic strain rates.

At this stage it is convenient to introduce the dimensionless variables

$$\begin{aligned} &x = \sigma/\sqrt{3}\,\tau_0, \qquad y = \tau/\tau_0 \\ &e = \sqrt{3}\,\varepsilon G/\tau_0, \qquad g = \gamma G/\tau_0 \end{aligned} \tag{5.2.6}$$

and $s = (x^2 + y^2)^{1/2}$. A characteristic time $t_c = \beta\tau_0/G$ may be identified and in what follows primes will imply differentiation with respect to the dimensionless time $T = t/t_c$. In terms of these variables (5.2.1), (5.2.2) and (5.2.3) take the form

$$e' = x' + \begin{cases} 0; & s < 1 \\ (xe' + yg')x; & s = 1 \\ x/s; & s > 1 \end{cases} \tag{5.2.7}$$

$$g' = y' + \begin{cases} 0; & s < 1 \\ (xe' + yg')y; & s = 1 \\ y/s; & s > 1 \end{cases} \tag{5.2.8}$$

$$\lambda = (xe' + yg')/2\beta\tau_0 \tag{5.2.9}$$

and the specifications of the problem are $g' = \alpha$, $e' = 0$ with $x(0) = 1$, $y(0) = 0$, $e(0) = 1$, and $g(0) = 0$. If, since $\dot{\gamma} > 1/\beta$, we assume that the material is in the rate dependent regime (i.e. $s > 1$) for T near $T = 0$ then (5.2.7) and (5.2.8) take the form

$$\begin{aligned} x' + x/s &= 0 \\ y' + y/s &= \alpha \end{aligned} \tag{5.2.10}$$

which lead to

$$ss' + s = \alpha y. \tag{5.2.11}$$

At $T = 0$, $s = 1$, $y = 0$ and it follows that $s'(0) = -1$ requiring that $s < 1$ for $T > 0$ and thus contradicting the assumption that the rate dependent equations apply. If we consider the response to be elastic ($s < 1$) we have $s = (1 + \alpha^2 T^2)^{1/2}$ contradicting this assumption. The rate independent equations when applied to this situation take the form

$$\begin{aligned} x' + xy\alpha &= 0 \\ y' + y^2\alpha &= \alpha \end{aligned} \tag{5.2.12}$$

the solution of which with $x(0) = 1$, $y(0) = 0$ is

$$\begin{aligned} y &= \tanh g \\ x &= 1/\cosh g. \end{aligned} \tag{5.2.13}$$

This solution will apply up to the time at which the equivalent plastic shear strain rate $2\sqrt{(\frac{1}{2}\dot{e}_{ij}{}^p\dot{e}_{ij}{}^p)}$ achieves the value of $1/\beta$.

For the Prandtl–Reuss equations in the general case we have $\dot{e}_{ij}{}^p = \lambda s_{ij}$, $\lambda = s_{ij}\dot{e}_{ij}/2\tau_0{}^2$ from which it follows that

$$\tfrac{1}{2}\dot{e}_{ij}{}^p\dot{e}_{ij}{}^p = \lambda^2\tau_0{}^2 = (s_{ij}\dot{e}_{ij})^2/4\tau_0{}^2 \tag{5.2.14}$$

and

$$\dot{\gamma}_{eq}{}^p = s_{ij}\dot{e}_{ij}/\tau_0. \tag{5.2.15}$$

In combined tension and torsion this takes form $(\sigma\dot{e} + \tau\dot{\gamma})/\tau_0$ and in the present case this is $\tau\dot{\gamma}/\tau_0$, which in dimensionless form is αy. Thus the character of the deformation will change when $y = 1/\alpha$, i.e. when $g = g^*$ given by

$$g^* = \tfrac{1}{2}\ln[(\alpha + 1)/(\alpha - 1)]. \tag{5.2.16}$$

The time at which this occurs is $T^* = g^*/\alpha$.

For $T > T^*$ the rate dependent equations apply subject to the conditions at $s = 1: x^* = (\alpha^2 - 1)^{1/2}/\alpha$, $y^* = 1/\alpha$. After differentiation and some manipulation, y can be eliminated from equation (5.2.11) to give an equation for s in the form

$$(ss')' + 2s' = \alpha^2 - 1. \tag{5.2.17}$$

From (5.2.11) the condition on s' at $T = T^*$ is $s'(T^*) = 0$. On integration of (5.2.17) with this initial condition we obtain

$$ss' + 2s = (\alpha^2 - 1)(T - T^*) + 2. \tag{5.2.18}$$

We have not been able to obtain a closed form solution for this equation although it is easily integrated numerically for any particular value of α. When $s(T)$ has been obtained $x(T)$ is then computed by constructing numerically

$$u(T - T^*) = \int_{T^*}^{T} (1/s)\, dt \tag{5.2.19}$$

from which

$$x(T - T^*) = x^* e^{-u(T-T^*)}; \qquad T > T^*. \tag{5.2.20}$$

Solutions obtained in this manner for various values of α are shown in Fig. 3.

Since the example is illustrative the general trend of the results is more useful than the details and an approximate solution which has the correct initial and asymptotic behavior is

$$s \approx 1 + r(T - T^*) \tag{5.2.21}$$

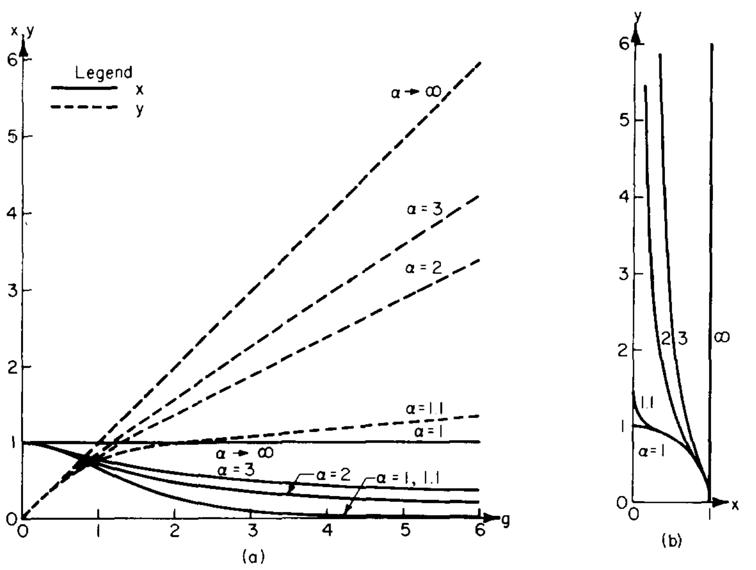

Fig. 3. Solution to the problem of a tube initially stretched to yield in tension and then twisted at a constant rate. (a) Axial and shear stress parameters vs rotation for various twisting rates. (b) Shear stress parameter vs axial stress parameter for various twisting rates.

where r, by substitution into (5.2.18) and comparison of terms in $(T - T^*)$, is given by

$$r = \alpha - 1 \tag{5.2.22}$$

from which

$$x \approx x^*/[1 + r(T - T^*)]^{1/r} \tag{5.2.23}$$

and

$$y \approx 1 + r(T - T^*). \tag{5.2.24}$$

The interesting implications of this solution are these. In the $x - y$ plane (Fig. 3b), the solution point moves around the circle and then leaves it tending asymptotically to the y-axis. With reference to the approximate equations and to Fig. 3(a), the solution for $x(T)$ is similar to that for the Prandtl–Reuss material but tends less rapidly to zero; that for $y(T)$ is quite different tending to the line $g(\alpha - 1)/\alpha$. For large values of α we have $y \approx g$ and $x \approx 1$ corresponding to an elastic solution. It is also interesting to note that there is no range of α for which the solution is entirely viscoplastic; the Prandtl–Reuss equations apply for a finite initial time for all finite values of α. When they do not, the solution is entirely elastic.

(iii) *Shock loading*

Although these previous examples lead to some insight as to the deformation produced when the limiting velocity is achieved they are somewhat in the nature of mathematical exercises due to the neglect of inertia terms. As an example of the influence of the limiting velocity in a situation where inertia terms can be included and the problem remains simple enough to allow an elementary solution we will consider the case of a one-dimensional plane wave through a material represented by the model of equation (4.2) and in particular illustrate the effect by computing the shape and thickness of a wave propagating without change of shape.

The study of the plastic shock wave has been given in much detail by several authors, for example, Rice, McQueen and Walsh[9] or Duvall[10]. The wave referred to as the plastic shock is the second component in the double wave structure which exists for most metals for impact pressures below a certain maximum. The first wave of this double wave is usually referred to as the elastic precursor and has the velocity of sound waves in the material. The velocity of the plastic shock increases with pressure and at pressures, at and above, that at which the velocity is equal to the precursor velocity only a single wave exists.

On the basis of a rate independent model for the plastic response, the plastic shock is a true shock in the sense of being a discontinuity in pressure and particle velocity. When a viscoplastic model is used, as was done for example by Kelly and Gillis[11], or Kelly[12], the plastic wave develops a structure and is not a true shock in the above sense although it is convenient to retain the term, and to use the term shock structure to describe the transition region of the wave from the initial level of pressure to the final level. For the viscoplastic as for the plastic model the elastic precursor is present when the pressure is below a certain level. When the pressure is such that the plastic wave speed is equal to or above the elastic wave speed a single structured wave initiating with a true shock is produced.

For the typical experimental configuration used to study shock propagation through solids, the assumption that the shock is propagating without change of shape, i.e. a steady-state wave exists, is essentially a computational convenience. Nevertheless, it yields a fairly large amount of useful information. For example, this procedure has been used previously by Johnson and Barker[13] to successfully compare with experimental shock profiles in 6061–T6 aluminum. It has also been used previously by us[11, 14].

In order that a steady-state structured shock exists it is necessary that the material model includes some strain rate sensitivity and that it includes some non-linearity in the relationship between pressure and density. It is essential then to supplement equation (4.2) by an equation relating the pressure p and density ρ which, since we will confine attention to compressional waves, will be concave upwards in the p, ρ plane, i.e. $\mathrm{d}p/\mathrm{d}\rho > 0$, $\mathrm{d}^2p/\mathrm{d}\rho^2 \geq 0$, $\rho \geq \rho_0$ where ρ_0 is the initial density.

For the purpose of this analysis we will take the material coordinate axis X_1 to be directed normal to the wave front and in the direction of propagation. The strain tensor ε_{ij} reduces to $\varepsilon_{11} = \varepsilon$ and $\varepsilon_{ij} = 0$ otherwise; and ε is related to the density through

$$\varepsilon = \rho_0/\rho - 1. \tag{5.3.1}$$

Also we take $\sigma_{11} = \sigma$, $\sigma_{kk} = -3p$, and denote X_1 simply by X.

The pressure density relationship will be taken in the form

$$p = -K\varepsilon + f(\varepsilon) \tag{5.3.2}$$

where K is the elastic bulk modulus and $f(\varepsilon)$ is such that $f(0) = 0, f'(0) = 0, f'(\varepsilon) \leq 0$ for $\varepsilon < 0$, and $f''(\varepsilon) > 0$ for $\varepsilon < 0$, where primes denote derivatives with respect to ε. In this case equation (4.2) reduces to

$$\begin{aligned} &2\dot{\varepsilon}/3 = (\dot{\sigma} + \dot{p})/2G; && |\sigma + p| < \tau_0 \\ &0 < |2\dot{\varepsilon}/3 - (\dot{\sigma} + \dot{p})/2G| < (1/\sqrt{3}\,\beta); && |\sigma + p| = \tau_0 \\ &2\dot{\varepsilon}/3 = (\dot{\sigma} + \dot{p})/2G + (1/\sqrt{3}\,\beta)\mathrm{sgn}(\sigma + p); && |\sigma + p| > \tau_0 \end{aligned} \tag{5.3.3}$$

where sgn denotes the signum function.

As noted above we direct our attention here to the two wave structure. The plastic wave is moving in the positive direction without change of shape with Lagrangean velocity V with respect to the undeformed material configuration X. The velocity V is bounded above by the elastic wave speed $[(K + 4G/3)/\rho_0]^{1/2}$ and below by the velocity $[K/\rho_0]^{1/2}$. All quantities such as stress $\sigma(X, t)$, pressure $p(X, t)$ and strain $\varepsilon(X, t)$ are functions of the time-like variable $\bar{t} = X/V - t$. The wave is imagined to have been produced by an impressed stress σ^* or strain ε^* at $\bar{t} \to -\infty$.

Either σ^* or ε^* is assumed to be large enough to exceed the elastic limit thus producing an elastic precursor but small enough that the plastic wave velocity is less than the velocity of the precursor. The field is thus the entire real line. The conditions as $\bar{t} \to \infty$ are those behind the precursor, namely, the stress σ_y, strain ε_y and pressure p_y given by

$$\begin{aligned} \sigma_y &= -(1 + 3K/4G)\tau_0 \\ \varepsilon_y &= -(3/4G)\tau_0 \\ p_y &= (3K/4G)\tau_0 \end{aligned} \tag{5.3.4}$$

these results being derived from the elastic equations neglecting $f(\varepsilon_y)$ in comparison with $-K\varepsilon_y$. Similarly since the strain rate tends to zero as $\bar{t} = -\infty$ the conditions there can be obtained by setting

$$\sigma + p = -\tau_0. \tag{5.3.5}$$

If we suppose that $\varepsilon = \varepsilon^*$ is specified, then as $\bar{t} \to -\infty$

$$\begin{aligned} p \to p^* &= -K\varepsilon^* + f(\varepsilon^*) \\ \sigma \to \sigma^* &= -(\tau_0 + p^*). \end{aligned} \tag{5.3.6}$$

Conservation of mass and conservation of momentum[12] lead to the following relation between stress and density at any two points $\bar{t}_1$ and $\bar{t}_2$

$$\sigma(\bar{t}_2) - \sigma(\bar{t}_1) = \rho_0 V^2[\rho_0/\rho(\bar{t}_2) - \rho_0/\rho(\bar{t}_1)]. \tag{5.3.7}$$

In particular if we take $\bar{t}_1 \to \infty$ and $\bar{t}_2$ as any finite $\bar{t}$ this becomes

$$\sigma - \sigma_y = \rho_0 V^2(\varepsilon - \varepsilon_y) \tag{5.3.8}$$

from which the velocity V is given in terms of ε^* by

$$V = [(1/\rho_0)(\sigma^*(\varepsilon^*) - \sigma_y)/(\varepsilon^* - \varepsilon_y)]^{1/2}. \tag{5.3.9}$$

From this equation it is clear that V is bounded below by $(K/\rho_0)^{1/2}$.

The deviatoric equations (5.3.3) can be written in terms of $\bar{t}$ on noting that the meaning of $(\cdot)$ is $\partial(\ )/\partial t_{x=\text{const}}$ which equals $-\mathrm{d}(\ )/\mathrm{d}\bar{t}$, and that in compression $(\sigma + p) < 0$. Thus to compute the wave profile for $\varepsilon(\bar{t})$ we use

$$(4G/3)(\mathrm{d}\varepsilon/\mathrm{d}\bar{t}) = \mathrm{d}(\sigma + p)/\mathrm{d}\bar{t} + \begin{cases} 0; & -(\sigma + p) < \tau_0 \\ \text{indeterminate}; & -(\sigma + p) = \tau_0 \\ 2G/\sqrt{3}\,\beta; & -(\sigma + p) > \tau_0 \end{cases} \tag{5.3.10}$$

with σ related to ε through equation (5.3.8) and p to ε through (5.3.2).

The solution is in three parts. For a region of the real line which (since the origin $\bar{t} = 0$ is arbitrary) we may take to be $0 < \bar{t} < \infty$ the solution is the uniform state $\sigma = \sigma_y$, $\varepsilon = \varepsilon_y$, $p = p_y$. For $\bar{t} < 0$ integration of equation (5.3.10) gives

$$(4G/3)(\varepsilon - \varepsilon_y) = (\sigma - \sigma_y) + (p - p_y) + (2G\bar{t}/\sqrt{3}\,\beta) \tag{5.3.11}$$

The profile $\varepsilon(\bar{t})$ is thus given by the solution of

$$(K + 4G/3 - \rho_0 V^2)(\varepsilon - \varepsilon_y) - f(\varepsilon) + f(\varepsilon_y) = 2G\bar{t}/\sqrt{3}\,\beta \tag{5.3.12}$$

which can be solved readily for any selected function $f(\varepsilon)$. This solution applies for values of $\bar{t} > \bar{t}^*$ where $\bar{t}^*$ is given by $\sigma + p = -\tau_0$, i.e. for $\varepsilon = \varepsilon^*$, beyond which a uniform solution applies, viz. $\varepsilon = \varepsilon^*$, $\sigma = \sigma^*$, $p = p^*$, for $\bar{t} < \bar{t}^*$. Noting that $\sigma_y + p_y = -\tau_0$ and that $\sigma^* + p^* = -\tau_0$ equation (5.3.11) gives as the value of $\bar{t}^*$

$$\bar{t}^* = (2\sqrt{3}\,\beta/3)(\varepsilon^* - \varepsilon_y). \tag{5.3.13}$$

When $\rho_0 V^2 = K + 4G/3$ the solution for $0 < \bar{t} < \infty$ is $\varepsilon = 0$; for $\bar{t} = 0$, $\varepsilon = \varepsilon_y$; for $t^* < t < 0$, $\varepsilon(t)$ is given by equation (5.3.12) as before and t^* by equation (5.3.13).

Cases where the pressure exceeds this can be treated in a similar way. However, the stress at the wave front will depend on the form of pressure volume relation selected. Moreover, in a real material, the pressures for which this would occur, are such that the neglect of thermal effects, in particular the influence of temperature on the dislocation motion, would be unacceptable.

In Fig. 4 are shown several wave profiles calculated on the basis of equation (5.3.12) using the parameters $3K/4G = 2$ and $3\tau_0/4G = 0{\cdot}005$ which correspond approximately to the aluminum tests reported by Johnson and Barker[13]. For these examples the function $f(\varepsilon)$ was taken to be $3K\varepsilon^2/2$. Results are plotted in the non-dimensionalized form $-\sigma/\tau_0$ vs $\bar{t}/\beta$ for several impact intensities, specified by the plastic wave speed parameter $\rho_0 V^2/K$. Figure 4 indicates a substantial disparity between the calculations and observed dependences of wave thickness on shock intensity. The experimental observation that shock thickness decreases with increasing impact intensity is clearly opposite to the trend of the calculations.

An important feature in the development of a steady wave profile is the competing effects of material nonlinearity in its pressure, volume relation and dissipative effects, in this case the viscoplasticity. Examination of the former with regard to possible selection of a more suitable function $f(\varepsilon)$ was fruitless. It can easily be shown that no pressure–volume relation satisfying the prescribed conditions can significantly alter the results shown in Fig. 4.

Thus we look to the viscoplastic relation to attempt to improve the computed results. Previous calculations of shock thicknesses based upon more complex dislocation models[11] showed a clear decrease of thickness with increasing shock pressure. Hence, it appears that the simplifications incorporated in equation (4.2) have eliminated an aspect of material behavior of some significance to the present problem. The main experimental observations relating to dislocation flux omitted from (4.2) are the increase of dislocation density with increasing deformation and a definite range of stress over which the dislocation velocity undergoes the transition from near zero to near maximum. Effects of a transition stress range were investigated for some simple problems in our initial paper[2] on ideal viscoplasticity. In a later paper[15] effects of dislocation multiplication were used to describe the upper yield point phenomenon.

Each of these effects was incorporated separately into the foregoing plastic wave equations to see whether calculated results could be made to exhibit a decreasing shock thickness with impact intensity. Dislocation multiplication was assumed to increase the available dislocations (hence the plastic rate) in direct proportion to the accumulated plastic strain. The corresponding form of equation (5.3.10) is not directly integrable as in the simple case

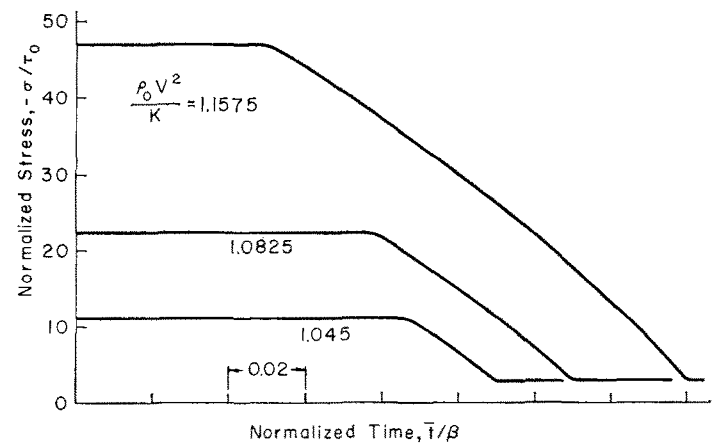

Fig. 4. Calculated stress profiles for three steady-state waves. Basis of calculation is specified in text; constants used approximate those appropriate for 6061–T6 aluminum.

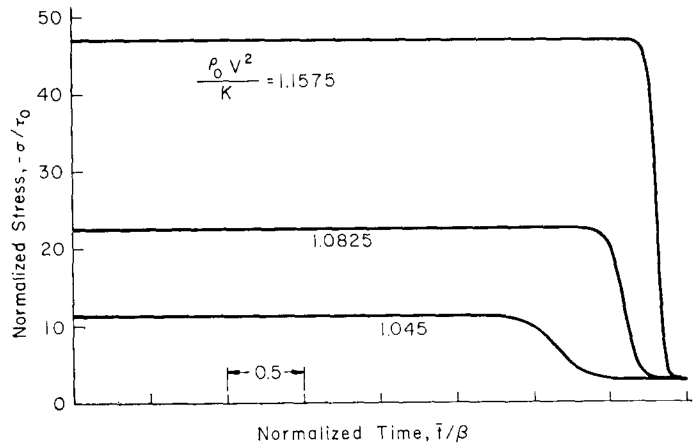

Fig. 5. Stress profiles for three steady-state waves calculated by numerical integration of a plastic strain rate relation incorporating a transition stress range as described in text.

considered initially. However, numerical solutions are easily obtained using a computer as the equation is only first order. Various ratios of initial dislocation density to multiplication coefficient were used but all results continued to show an increase of shock thickness with intensity. Thus, we conclude that dislocation multiplication is not a crucial aspect of material behavior in the present situation. The main effect of multiplication seems to be a change in scale of the normalized time, $\bar{t}/\beta$. Increasing the number of dislocations decreases the effective average relaxation time in comparison to β which is fixed. The time for transmission of the wave profile then becomes reduced by approximately the same factor as the dislocation density is increased.

A transition stress range was investigated next. We assumed a linear increase in plastic strain rate from zero at $|\sigma + p| = \tau_0$ to the prescribed maximum value at $|\sigma + p| = \tau_1 > \tau_0$. Again, the resulting form replacing equation (5.3.10) does not yield a closed form solution but can be integrated numerically with relative ease. This feature seems to provide the desired results: a decrease of shock thickness with impact intensity. Figure 5 shows results calculated using the previously specified values as for Fig. 4 and taking $\tau_1 = 2\tau_0$ as defining the transition stress range. Here the gradual increase of plastic strain rate with stress reduces the average relaxation time in comparison with the fixed parameter β and increases the normalized time for transmission of the wave profile.

Examination of the numerical results underlying Fig. 5 shows that plastic relaxation proceeds at rates substantially lower than the maximum. In other words, the driving stress does not reach a magnitude of τ_1 for the range of impact intensities studied here. In relation to the simple material model of equation (5.3.10) the implication is that most of the action should be made to occur at the point currently labeled "indeterminate," viz. $|\sigma + p| = \tau_0$, if any further attempt is made to improve the correspondence of this model to reality.

6. CONCLUDING REMARKS

The concept of ideally viscoplastic crystalline solids arises from two physical features of dislocation motion which fix an upper bound on the plastic strain rate. These are a saturation value for dislocation density, and a limiting value for dislocation velocity. Combined, these place an upper limit on dislocation flux and, therefore, on rate of plastic deformation if caused mainly by dislocation motion, as is the usual case for polycrystalline metals. The idealization of this flux upper bound consists is postulating a simple transition function from zero flux to the maximum. This selection of a function, its variables and parameters can be guided by experimental results from dislocation observations and the character of the problem under investigation.

At its present stage of development the theory of ideally viscoplastic materials has only limited usefulness. In ordinary metals at ordinary strain rates the limiting value of dislocation flux is not achieved. Thus, the actual maximum flux, or plastic strain rate, must be interjected as a known parameter into theoretical solutions for them to have practical significance. Alternatively, a more complex transition function may increase the applicability of the theory to situations of practical importance. Meanwhile, we have shown here an interesting limiting case, of academic interest, but useful in giving some insight into material behavior and what factors influence it.

One important suggestion in applications of the concept of ideal viscoplasticity is that some dislocation parameters seem to have little effect on certain aspects of mechanical response unless they are changed by several orders of magnitude. Hence their accurate functional representation is probably unimportant in examining those aspects of the response.

In this spirit we postulated an extremely simple transition function, an abrupt change from zero to maximum flux at a critical stress. This was easily formulated into a three-dimensional stress, strain, strain rate relation which was illustrated by application to three example situations. Exact closed-form solutions were found in two of these cases and a very good approximate solution in the other. We believe that the substantial utility of closed-form solutions, even when they are known to be of an approximate nature, indicates a great potential usefulness for such idealizations.

The physical feature mainly emphasized by our choice of transition function and examples is the effect of a limiting dislocation velocity. The mathematical results obtained indicate that a dual response ensues. The material behavior was rate sensitive during only a portion of each problem examined. On this basis we expect a real material to be significantly rate sensitive during corresponding portions of real physical investigations and approximately rate insensitive otherwise.

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Абстракт—Некоторые физические результаты, возникшие из исследования движений дислокации, приводят к понятию идеальной вязкопластичности. В частности, существование ограниченной скорости дислокации, сопряженной с верхным пределом плотности дислокации, определяет верхную границу потока дислокации. Таким образом, во многих интересных случаях, скорость пластической деформации в поликристаллических материалах должна также быть ограничена. Можно идеализировать это физическое состояние, путем постулирования функций нарастания от нуля до максимального значения потока, как это можно несложным способом, согласающимся с исследованной задачей и, далее, рассматривать особенности решения. Дается, здесь, простая функция нарастания, которая приводит к умеренной простоты зависимостям многоосных напряжений, деформации и скорости деформации. Эти зависимости иллюстрируются применением к отдельным примерным задачам. Общим признаком решений исследованных примеров является факт, что поведение материала частично зависит от скорости и частично не зависит. Это означает, что соответствующие физические состояния изображены наличием больших потоков дислокации в некоторый момент времени и очень малых потоков для других периодов времени.